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HNRS Distinction Project
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Mathematical Compositions and Musical Calculations

Introduction:

Math is a wonderfully complex enigma. It has so much to offer the world, but the world openly hates every mention of its name. To truly understand math is to accept it and take the time to get to know it properly. But no one has the time for that so here we are.

Music is simple. It makes people feel every emotion on the spectrum and we still push for more. It moves us, shapes us, and surrounds us. The world takes the time to understand music because the world sees it as something worth knowing. People pay attention to music as a pastime and an addition to daily life. Music does not take brain power to understand its purpose.

Until now.

Music is a medium with which we present our world and I am here today to use it to explain mathematics. Fractals, Chaos Theory, Fourier Series, Fibonacci's Sequence, Modular Arithmetic, and Symmetries can all be expressed through music. This music can then be used to explain those concepts. The purpose of this paper is to explain how music can explain these complex topics and, hopefully, to increase your knowledge on mathematics, music, or both. Math is a big beautiful world and I am here to show it to you.

Fractals:

Fractals provide good screen savers, artwork, and lyrics in Disney songs. Some say that a, "fractal is a never-ending pattern" (Fractal Foundation). This means that there is a pattern in place that the shape must follow so that as the shape is enlarged, more of the fractal is revealed. A perfect fractal goes on to infinity and this is what makes it so intriguing. The deeper you look at a fractal the more of it you can see since for a fractal to exist you must be able to look at just part of the fractal and see the entirety of it at the same time. This is an odd concept; however, it makes much more sense while examining a fractal.

The precise definition of a fractal is difficult to determine, because there is not a single definition that is agreed upon. Mandelbrot is revered in the field of fractals and he himself has put out definitions but later recalled them since they were not specific enough for his needs (Feder 2). The most agreed upon part of the definition is that a fractal is a repeated pattern on different scales (Fractals).

Self-similarity, scaling, and dimension are critical to the way that fractals are viewed. Self-similarity is the idea that the closer a fractal is viewed the same shape continues to emerge (Liebovitch 8). This is the pattern repeating itself smaller and smaller as it goes on forever. In order to view these small patterns, scaling is used: relating the size of the shape to the size of the pattern around it which is known to be larger (Liebovitch 8). The scaling factor is the value of the length of the old sections of the fractal in relation to the new pieces. This seems a bit ridiculous: why would we need to know the size of the smaller part of the fractal? The answer is that since all parts of the fractal look identical, there is a need to make sure that it is understood which part of the fractal is being viewed. It also helps to know if someone is referring to the fractal as a whole or a smaller part of the fractal. Dimension takes this scaling information and uses it to determine which parts of the fractals are new by taking the natural logarithm of the number of new pieces of the fractal over the natural logarithm of the scaling factor, $\ln(\#\ of\ new\ pieces)\ /\ln\ (scaling\ factor)$ (Liebovitch 8). The idea behind dimension is referred to as box counting where a "box" must have greater dimension than the fractal in question, and the dimension is the count of how many boxes it takes to cover the complete fractal (Chapter). This concept can be seen in Figure 1.1 as a fern that acts as a fractal is covered in boxes and the scales show the smaller parts of the fractal. Self-similarity, scaling, and dimension are how to view fractals so that they can be understood as a whole.

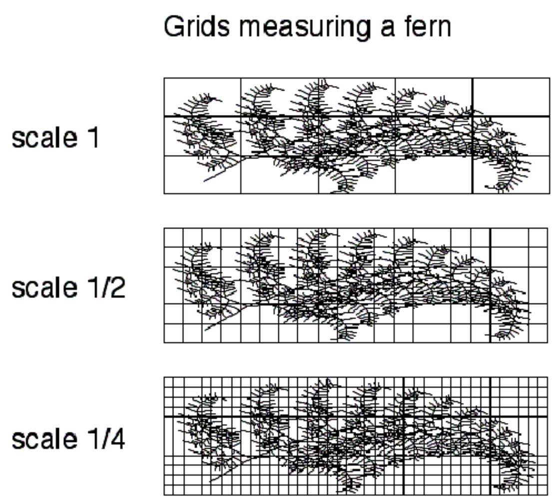

Figure 1.1 Box Counting (Chapter)

One famous fractal is Koch's snowflake which is traditionally started by an equilateral triangle. However, each individual side behaves the same, so it is easier to discuss the fractal as Koch's curve. The simplest explanation of the curve is that the line begins as three units long, then the middle unit is replaced by two lines with each of the lines equal to one unit of length (Liebovitch 54). These two lines add a point over the middle third. This pattern continues and with each iteration every straight line gets a point in the middle third. Every iteration is given a number, for example the original line is n=0, after the first point is added it is n=1 and so on every time that the points are added to all of the existing straight lines (Feder 16). The snowflake is the curve just on all three sides of the equilateral triangle expanding on all sides simultaneously. Koch's curve and snowflake therefore looks like the images in figures 1.2 and 1.3.

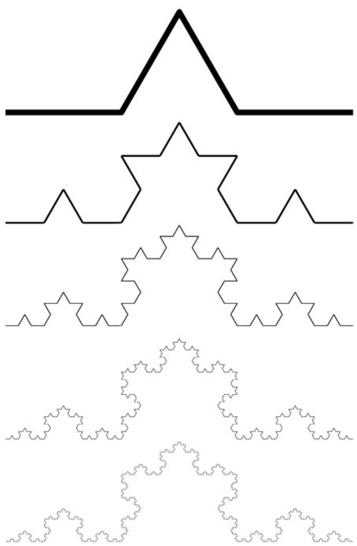

Figure 1.2 Koch's Curve (Julie)

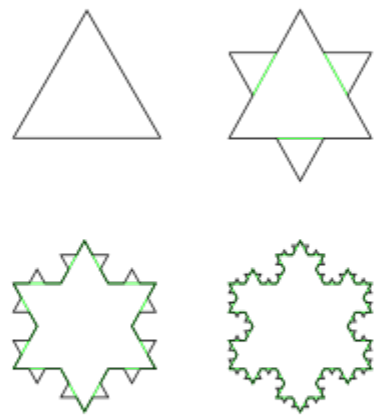

Figure 1.3 Koch's Snowflake (Wikipedia)

From a mathematical point of view, Koch's snowflake is intriguing. Due to the points being added at every iteration, Koch's snowflake has infinite perimeter that has a self-similarity dimension (Liebovitch 54). The infinite perimeter comes from the always expanding sides and the self-similarity dimension finds the perimeter's dimension and we calculate this by dividing the natural logarithm of new pieces (4) by the natural logarithm of scaling factor (3). This comparison of 4/3 also gives the amount that the perimeter grows during each iteration. This means that for each iteration the current perimeter is multiplied by 1.33 to get the new value of the perimeter.

The Sierpinski Triangle is another famous fractal that is studied because of the simplicity of the construction combined with its mathematical aspects. The Sierpinski triangle is created by taking an equilateral triangle and marking points exactly halfway between each of the sides and connecting them (creating an upside down triangle), this idea is repeated to create the fractal (Parsons). The result of the first iteration is 3 smaller triangles within the first one on each side of the upside-down triangle. This continues with three smaller triangles within each of those smaller triangles and the pattern continues. The shape is formed around these upside-down triangles that are "removed" from the shape. This means that there are no additional triangles found in the upside-down triangles. The shape quickly complicates itself and it is easy to see the fractal being produced in Figure 1.4. No triangles are added to the spaces that have been removed, but there will always be more space to create new triangles within the remaining triangles.

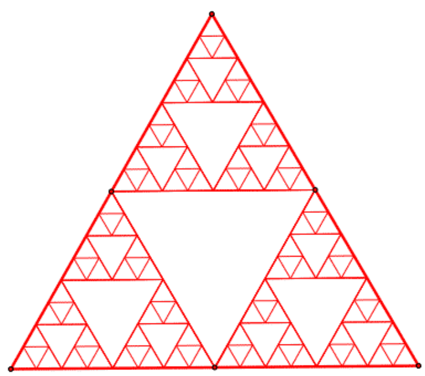

Figure 1.4 The Sierpinski Triangle (Parsons)

The first piece of fractal music that was written for this project is a basic musical fractal. It starts as six notes: a quarter note followed by four eight notes and another quarter note. At each iteration the longest notes are replaced by a smaller version of the same pattern. As you listen, there are long tones to designate when the next iteration is about to begin. This is a simple fractal meant to demonstrate the concept of a shape that continues to become more and more complex. The piece is a complete fractal because once the most complex iteration is heard, the piece begins to simplify to make the piece symmetric. This confirms that the piece itself is a fractal that grows more complicated in the middle, much like Koch's snowflake and the Sierpinski Triangle.

The second piece is a representation of Koch's Snowflake up to iteration n=3. Between iterations there is three beats of rest so that the listener can understand how much more complicated the piece becomes as n increases. The piece creates the fractal as can be seen visually from the sheet music as the ups and downs of the music match the idea of triangles being added. The longer notes represent the straight lines of the fractal and so in each iteration the long notes were broken up and notes were added in the center to create the feel of a triangular point.

The final fractal piece is a representation of the Sierpinski Triangle. We begin with the bottom line of the triangle: notes are added halfway between the previous notes until there is very little rest in the measure. The rests correspond to the triangles being removed from the larger triangle. The music travels along the bottom of the triangle as described followed by the left side and then the right. The sides are constructed similarly only with rising and falling pitches to match the idea of a triangle. The first measure of each section begins with the end points of the triangles. The result is a building of each side of the triangle separately and then combined for the final line of the song.

Chaos Theory:

Chaos theory appears to be a random collection of points and lines; however, it is completely the opposite. Chaotic functions are nonlinear functions that are practically impossible to predict (Fractal Foundation Chaos Theory). This leads to graphs and figures that are disjointed and as such they are given the name chaotic.

The main difference between chaos and randomness is the deterministic factor of chaos. This means that the system has order even if it is difficult to see. Chaotic systems are recursive meaning that each data point was found by using the point value before it. These connections are rarely obvious and that is why people assume that these points were created from randomness (Liebovitch 118). Chaotic equations are interesting because they are sporadic, and they are greatly affected by seemingly minute differences. Some of these small changes happen when different values are used as the first value of a function, also called initial conditions. Initial conditions are a large part of this because one of the definitions of chaos is that, "if a system is rerun with almost the same starting conditions, the values of the variables measured at the same time of the two runs separate from each other exponentially fast as a function of time" (Liebovitch 168). This explains why when initial conditions are off by small amounts, the end result is different from the original by a noticeable amount. Many chaotic equations are shown side by side with other cases that have nearly identical initial conditions, because this shows the extent of the chaos. The most famous chaotic idea is the butterfly effect which states that if a butterfly flaps its wings on one side of the world it will affect if a tornado occurs or not somewhere else in the world (Chaos 2019). The butterfly flapping its wings is the initial condition and the tornado or lack thereof is the exponentially different result that can occur from this scenario. Thus that one small difference of the butterfly's wings changed the recursive relationship of the Earth's climate to create a tornado or not.

The equation x(n+1)=3.95[x(n)][1-x(n)] is chaotic because it is deterministic, as can be seen by the multiplication of the previous term in the equation. The equation begins by an arbitrary choice of a starting point. Table 2.1 shows the equation with three different starting values: 0.892, 0.893, and 0.894. The Table 2.1 shows the journey of the equation with all three of the different initial conditions.

x(n)=1	0.892	0.893	0.894
x(n)=2	0.380	0.360	0.374
x(n)=3	0.931	0.910	0.925
x(n)=4	0.253	0.324	0.274
x(n)=5	0.747	0.865	0.786
x(n)=6	0.746	0.461	0.664
x(n)=7	0.748	0.981	0.881
x(n)=8	0.745	0.074	0.414
x(n)=9	0.750	0.271	0.958
x(n)=10	0.741	0.780	0.159
x(n)=11	0.758	0.678	0.528
x(n)=12	0.725	0.862	0.984
x(n)=13	0.788	0.470	0.062
x(n)=14	0.660	0.984	0.230
x(n)=15	0.886	0.062	0.700
x(n)=16	0.399	0.230	0.830

Table 2.1 Table of the chaotic outputs of the given equation (Equation and first column of values began in Liebovitch's book.)

These initial conditions differ by 0.001 and they all follow the same pattern, but they take off in very different directions and end at points that differ by at least 0.169. The musical piece created from these values shows these situations by each number being associated with a note on the staff. Lower values correspond to lower notes and vice versa. It is possible to follow along with the piece by using Table 2.1. The music plays each column individually and then all three columns are played at the same time to help give perspective on the differences between the situations.

The equation x(n+1)=4 [x(n)][1-x(n)] is chaotic because the values are deterministic. When x[0]=0.4, Table 2.2 emerges to show the values of the chaotic equation:

x[0]	0.9600
x[1]	0.1536
x[2]	0.5200
x[3]	0.9984
x[4]	0.0064
x[5]	0.0254
x[6]	0.0990
x[7]	0.3568
x[8]	0.9180
x[9]	0.3011
x[10]	0.8418

Table 2.2 Table of the chaotic outputs for the equation x(n+1)=4 [x(n)][1-x(n)]

These values can then be applied to a song such as "Mary had a Little Lamb" by taking the total number of beats (32) and multiplying by the percentages given in the table and rounding to whole beats to select the notes. For each of these products that were found, the selected note was raised by one pitch (one note was chosen twice, so it was moved twice). Some measures were affected three times within the four beats, other measures were not touched. That is how chaos theory works, there is very little visible logic behind the results of these equations. And although the song sounds as if random notes are raised, the choices made followed a very precise chaotic pattern. The original song is played first followed by the chaotically adjusted version so the listener can hear the difference.

Fourier Series:

There are curves in mathematics that are very difficult to quantify. There are plenty of equations that are difficult to write an equation for: That is why we have Fourier Series. Fourier Series uses sines and cosines to interact with one another to create a harmonic analysis of the curve and to mimic the actual graph (Weisstein Fourier). These are not exact equations, but some of the predictions are fairly close to the desired end result.

Fourier Series is just that—a series. The basic equation is $f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$ where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\, dx$, and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\, dx$ (Weisstein Fourier). When multiple iterations have been achieved, the formulas produce a summation that is a large string of sines and cosines with coefficients abounding. The more iterations of the series that are used, the more accurately the prediction will be close to the actual graph. Each iteration adds curves to the prediction until it slowly matches with the intended picture; the original prediction is typically a horizontal line on the x-axis and as the series increases the prediction improves to shadow the graph more accurately (Weisstein Fourier). The more iterations that the series performs provides the series with more information and more terms which is what allows the series to become more accurate. These predictions are close, but they all include what is known as the Gibbs phenomena: this is when there is a slight bump on the prediction directly before the true values drastically change (Weisstein Gibbs). This is because sines and cosines make up the prediction and they are wave equation that do not keep up with big interval changes as well as other changes. These waves can also be seen throughout the graph; however, they are simply more distinct around large changes.

The Fourier music piece is based around the square curve and how the Fourier Series forms around that. The first two notes that you hear are the true square curve that the Fourier Series will try to replicate. Following a measure of rest is the first iteration of Fourier Series, then another rest and the second iteration, and so on. It is easy to hear how much closer the estimations grow toward the square curve. The major difference is that the two notes are disconnected in the square curve; however, Fourier connects the lines (which is easy to hear in the piece). It is also possible to hear the Gibbs phenomena, every iteration (following the first) overestimates the maximum and minimum of the original function due to this principle (Weisstein Gibbs). The musical piece follows Figure 3.1 since the black lines signify the square curve, the red line is the first iteration of Fourier Series, the orange line is the second iteration, followed by the yellow line, then the green line, and finally the blue line.

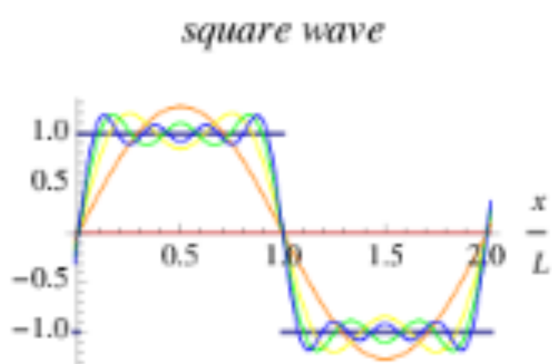

Figure 3.1 The square curve and the first five Fourier Series outputs attempting to replicate the initial curve

Fibonacci's Sequence:

1,1,2,3,5,8,13,21,34…Fibonacci's sequence is one of the most popular string of numbers that are discussed in class today. And it's not even Fibonacci's greatest contribution to math! However, the sequence explains many different aspects of life and once you have heard about it, you will not view anything the same again.

Fibonacci's sequence is the list of numbers where the next number is a sum of the previous two. Fibonacci invented the sequence as a riddle: a pair of baby rabbits can produce offspring after two months and then every month thereafter the pair produces a pair of rabbits one male and one female (Life 2017). The sequence accounts for each pair of rabbits counting as one and as the numbers grow and more rabbits are being produced that gives us the sequence that we know today. This occurs since the previous two numbers (of pairs of rabbits) will produce offspring.

When the sequence is taken and used graphically it can form an even more impressive value: the Fibonacci spiral. This is created by taking squares where the length of one side is the value of each of the numbers in the sequence and then these squares are built off of each other to form larger and larger rectangles built of the Fibonacci squares (Life 2017). The trick to making the rectangle work is to take the new square and line it up with the added length of the previous two squares. To turn this rectangle into a spiral a line is drawn from the inside corner of the innermost one-square and through its opposite corner and the curve continues to travel through opposite corners (Life 2017). This spiral is very famous in the math world because of all of the work that goes into creating it.

A similar spiral is created by setting four bugs lose to play an interesting game of tag. The bugs are set in an exact square and they all begin to move towards each other in the same direction at the same rate and as they move towards the bug to their right, they create spirals into the middle of the square (Fibonacci). This pattern can also be extended to different geometries; however, a square provides enough information. The proportion as the bugs move looks very familiar as it is the ratio between a Fibonacci number and its previous iteration (Fibonacci). This comparison is referred to as the Golden Ratio.

The Golden Ratio has a glorified name for a reason. The Golden Ratio is equivalent to 1.618 or $\frac{1+\sqrt{5}}{2}$ and it can be expressed in many ways as well since it is such a beloved value (Fibonacci). The Golden Ratio is created from Fibonacci's Sequence based on the ratio of the dimensions of the rectangle (length divided by width) that helps define Fibonacci's curve. This ratio and rectangle also appear in many famous works of art and throughout history: the Parthenon was built to the dimensions of the Fibonacci rectangle, the Mona Lisa has the essence of the spiral starting on her face, and Composition in Red, Yellow, and Blue also is drawn according to the ratio (Golden). It is said that the ratio is very pleasing to the eye and draws people in, which perhaps is why so many famous pieces of art replicate it.

Plants follow this pattern of numbers that were all created from a simple riddle! Flowers grow Fibonacci numbers of petals in spirals, pinecones do the same, and even the pineapple follows this pattern which has been deemed efficient for explaining characteristics of plants (Fibonacci). This means that nature even follows this sequence of numbers and makes it out to be the most important discovery that a mathematician has made. The sequence applies to all parts of life and we have not heard anything yet.

Fibonacci's Sequence is based on the principle that each new element in a sequence is the sum of the previous two. To apply this to music, each new measure is a combination of the two immediately before it. There are two examples of this within the piece separated by a measure of rest. The first version uses quarter notes in the first and second measure and as these notes are added together, they are adjusted to take up less time in the measure so that it is easier to determine when the next element of the sequence begins. The second version begins with eighth notes and complicates much more quickly, even as it follows the same guidelines as the first version.

Modular Arithmetic:

Whenever someone asks what day of the week it will be in eight days, what they are really having you do is modular arithmetic! They are asking you to take your week divide the days by seven and take the remainder (in this case one) and add that to the current day of the week and determine what day it will be. Say it is a Tuesday, in eight days will be a Wednesday. This is the same as if someone had asked what day of the week it will be tomorrow or what day of the week it will be in 15 days. This is more of what modular arithmetic is, it allows for numbers to be comparable in different modulos.

The idea of modular arithmetic is the division algorithm stating that a=bq+r where a and b are integers where b is greater than zero and r is between 0 and b (Gallian 3). This is the idea that any number can be broken down to form this equation where a is the number that we are investigating, q is the modulo (or mod) which is the number that is being divided by, b is the largest whole number of times that q can go into a, and r is the remainder. The day of the week example has an a value of eight, q is seven since there are seven days in a week, b is one since seven goes into eight once, and the remainder has r equal to one. Based on modular arithmetic numbers can be equivalent if they have the same r value in the same modulo, this is called an equivalence class (Gallian 18). This is why one, eight, and fifteen days from Tuesday are all Wednesdays: One, eight, and fifteen are in the same equivalence class in mod 7. We will also see modular arithmetic in figuring out what month it will be if we are asking more than 12 months away.

Music is another example of modular arithmetic being used in daily life. Music has eight note scales. They are represented by the letters A through G. This means that every eighth note is the same letter repeated. The note is eight tones higher, but the letter is signified as the same. These notes together sound like the same pitch only in different octaves. Modular arithmetic surrounds us in ways (like this) that we do very naturally, yet this function is also used in a lot of advanced math practices. Many proof techniques require that modular arithmetic be understood so that more advanced topics can be proved true. For example, modular arithmetic turns a normal set of numbers into cyclic groups which allows for proofs referring to groups to apply to certain sets of numbers that would not form a group otherwise (Weisstein Cyclic). But as this is abstract algebra, I digress.

Modularity is a piece that takes a basic scale and arpeggio and shows how the different modulos affect the range of the piece. The first time through the scale is played regularly, then in mod 8, mod 6, mod 4, and mod 2 with B flat being the first note in each modulos. This shows how different notes become equivalent in their equivalence classes because in mod 6 and mod 2 B flat and A are in the same equivalence class, but in mod 4 and mod 8 they are not. This will change because in mod 6 and 2, B flat has the same remainder, but not in mod 4 or 8.

Modular the Beautiful is the same concept except that the piece is more recognizable, and the mod is more complicated. America the Beautiful is played as written the first time through, then mod 8 and then mod 4. It is intriguing because the notes are still near the correct relationships to hear the song, but they are not the same sound. This makes sense because the relationships between the notes are the same, just separated by less space than normal.

Symmetries:

All polygons can be transformed a different number of times. Squares can be turned 90º four times to reach its original position. The same square can be flipped vertically, horizontally, and diagonally in both directions. This is eight symmetries. A triangle has six symmetries and a circle has infinite symmetries. These symmetries are part of what defines the shapes and the study of them proves very interesting.

Symmetries of a square are as follows: Rotation of 0º/360º, Rotation of 90º, Rotation of 180º, Rotation of 270º, Flipped Horizontally, Flipped Vertically, Flipped from the top right to the left bottom, and Flipped from the top left to the right bottom. These flips and rotations can then be combined. Table 6.1 below demonstrates all of the possible two-combination flips and rotations and their end result. The original square can be viewed in the top left where A, B, C, and D denote the corners of each square and then (following the symmetries being applied) where those corners end up. The top row action occurs first followed by the side column action.

Symmetries of a Square	Stay	R_{90}	R_{180}	R_{270}	F_H	F_V	$F_{Left\ top\text{-}Right\ Bottom}$	$F_{Right\ top\text{-}Left\ Bottom}$
Stay	AB DC	DA CB	CD BA	BC AD	DC AB	BA CD	CB DA	AD BC
R_{90}	DA CB	CD BA	BC AD	AB DC	AD BC	CB DA	DC AB	BA CD
R_{180}	CD BA	BC AD	AB DC	DA CB	BA CD	DC AB	AD BC	CB DA
R_{270}	BC AD	AB DC	DA CB	CD BA	CB DA	AD BC	BA CD	DC AB
F_H	DC AB	CB DA	BA CD	AD BC	AB DC	CD BA	DA CB	BC AD
F_V	BA CD	AD BC	DC AB	CB DA	CD BA	AB DC	BC AD	DA CB
$F_{Left\ top\text{-}Right\ Bottom}$	CB DA	BA CD	AD BC	DC AB	BC AD	DA CB	AB DC	CD BA
$F_{Right\ top\text{-}Left\ Bottom}$	AD BC	DC AB	CB DA	BA CD	DA CB	BC AD	CD BA	AB DC

Table 6.1 Symmetries of a square

Symmetries of a triangle are as follows: Rotation of 0º/360º, Rotation of 120º, Rotation of 240º, Flipped top to bottom left, Flipped top to the right bottom, and Flipped from the left bottom to the right bottom. Table 6.2 demonstrates all of the two-action combinations of the triangle with the top row action applied before the left column. The original placement is in the top left of the grid.

Symmetries of a Triangle	Stay	R_{120}	R_{240}	$F_{Top\text{-}Left}$	$F_{Top\text{-}Right}$	$F_{Left\text{-}Right}$
Stay	A BC	B CA	C AB	B AC	C BA	A CB
R_{120}	B CA	C AB	A BC	A CB	B AC	C BA
R_{240}	C AB	A BC	B CA	C BA	A CB	B AC
$F_{Top\text{-}Left}$	B AC	C BA	A CB	A BC	B CA	C AB
$F_{Top\text{-}Right}$	C BA	A CB	B AC	C BA	A BC	B CA
$F_{Left\text{-}Right}$	A CB	B AC	C BA	B CA	C AB	A BC

Table 6.2 Symmetries of a triangle

These symmetry groups are called dihedral groups represented D_4 for squares and D_3 for triangles, as the square symmetries are of order eight and the triangle symmetries are of order six because that is the number of symmetries for each (Introduction). The symmetry combinations are interesting because they still can only produce the initial symmetries. The table is filled with eight different possibilities for how the square can end up. Some symmetries have inverses that render the square in its starting position. For example, using a square, a 90º turn followed by a vertical flip is the same as a flip from the right top to the left bottom. The flips are their own inverses while the turns are a bit more complicated The symmetries encompass all eight possible outcomes for the square as long as the square returns to the same position as before (it would not be a symmetry if the square was turned to a diamond formation).

What is also interesting is that these groups are not abelian. Being abelian means that the elements in the group commute with one another (Weisstein Abelian). Multiplication is abelian because 2*3=3*2. Some symmetry elements are not commutative: a 90º turn followed by a vertical flip gets a very different result than a vertical flip followed by a 90º turn. Square symmetries can come in handy when trying to write proofs in generalities for all groups, because not all groups are abelian so the symmetries are a good counter-example.

The first symmetry piece was created using this concept by having four notes written on a square and then the same symmetries applied. The original notes are F, low F, E, and low G. I found that the music reacts in a similar way than that of the square's reflections and rotations. The music reads for each measure to be a square and then starting on the rows with two actions applied to them the table is read left to right. The result is intriguing to hear and to understand that it is a collection of eight different measures strategically placed and heard to understand how the square is also changed and moved.

The second piece is created in a very similar manner, only a triangle was used. Triangles have different symmetries than squares and the piece reflects this difference. The square reflections result in very different measures and notes while the triangle symmetries all stay within a certain range.

Interaction:

The purpose of this project, the research, and the musical pieces is to reach out to people and help show them some of the cool things that math can do. I am hopeful that this may spark an idea in someone, inspire someone to keep learning math, or to give a teacher a tool to help encourage their students. For these reasons the music and recordings are welcome to be played by anyone for any purpose as long as it encourages study in music or math.

I would like to encourage any teachers to try and incorporate these songs and descriptions into their classrooms. In only a few minutes a teacher can showcase one of these ideas to “wake up” their classroom and get more interaction from students. The pieces are written in such a way that even someone with neither a music nor a mathematics background should be able to comprehend what is being explained and played. For those who understand math, this project demonstrates a creative outlet that they can explore with mathematics of their choosing. Musicians similarly may feel more connected to the process and feel empowered to master math that is being displayed in such a way that appeals to them. Whatever the students’ background, this project was created to inspire intellectual inquiry.

Conclusion:

Music and math are the perfect combination because it allows logical people to see the beauty in music and the artist-minded people a chance of peeking into the wonders of math. More than that, music helps in the learning of math at all ages. In children, music and math combined classes produced two results: the first group showed longer attention spans and focus while they began to grasp the concept, while the second group enjoyed the experienced as it reinforced the learning (McDonel 53). This just goes to show that the two go hand-in-hand and provide a great way for everyone to find something they enjoy while they learn.

Fractals, Chaos Theory, Fourier Series, Fibonacci's Sequence, Modular Arithmetic, and Symmetries are mathematical topics that have stumped students for generations. I hope that in reading these explanations they have been made clearer. I also encourage you to go to https://mathmusic.pages.roanoke.edu/ and listen to the pieces being played. Math does not have to be scary, and music is not all fun and games.

Welcome to Mathematical Compositions and Musical Calculations.

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https://musopen.org/

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